

ON THE COBORDISM CLASSIFICATION OF MANIFOLDS WITH \mathbb{Z}/p -ACTION

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ABSTRACT. We refer to an action of the group \mathbb{Z}/p (p here is an odd prime) on a stably complex manifold as *simple* if all its fixed submanifolds have the trivial normal bundle. The important particular case of a simple action is an action with only isolated fixed points. The problem of cobordism classification of manifolds with simple action of \mathbb{Z}/p was posed by V. M. Buchstaber and S. P. Novikov in 1971. The analogous question of cobordism classification with stricter conditions on \mathbb{Z}/p -action was answered by Conner and Floyd. Namely, Conner and Floyd solved the problem in the case of simple actions with identical sets of weights (eigenvalues of the differential of the map corresponding to the generator of \mathbb{Z}/p) for all fixed submanifolds of same dimension. However, the general setting of the problem remained unsolved and is the subject of our present paper. We have obtained the description of the set of cobordism classes of stably complex manifolds with simple \mathbb{Z}/p -action both in terms of the coefficients of universal formal group law and in terms of the characteristic numbers, which gives the complete solution to the above problem. In particular, this gives a purely cohomological obstruction to the existence of a simple \mathbb{Z}/p -action (or an action with isolated fixed points) on a manifold. We also review connections with the Conner–Floyd results and with the well-known Stong–Hattori theorem.

1. INTRODUCTION

By a *stably (or weakly almost) complex manifold* we mean an orientable differentiable manifold with complex structure in its stable tangent bundle — the main object of study in the complex cobordism theory. The fixed point set of a smooth action of \mathbb{Z}/p on a differentiable manifold M (i.e. the fixed point set of a diffeomorphism of an odd prime period) is decomposed into the sum of connected fixed submanifolds of even codimensions (cf. [CF1]). The action of \mathbb{Z}/p induces a representation of \mathbb{Z}/p in the tangent space $T_q M$ at each fixed point $q \in M$. This representation is decomposed into the sum of non-trivial one-dimensional complex representations of \mathbb{Z}/p and the trivial representation in the tangent subspace to the fixed submanifold containing q . This decomposition is same for all fixed points within the same connected fixed submanifold. Each non-trivial one-dimensional representation of \mathbb{Z}/p is determined by an element $x_k \in (\mathbb{Z}/p)^*$, $x_k \neq 0 \pmod p$ (the generator of \mathbb{Z}/p acts by multiplication by $e^{2\pi i x_k / p}$). We refer to x_1, \dots, x_m as the *weights* of the \mathbb{Z}/p -action on M corresponding to the fixed submanifold (here m equals half the codimension of the fixed submanifold in M).

Definition 1.1. We refer to an action of \mathbb{Z}/p on a stably complex manifold M as *simple* if all the fixed submanifolds have trivial normal bundle. A simple action is called *strictly simple*, if the sets of weights for the action are identical for all fixed submanifolds of the same dimension.

The good example of a simple action is an action with only isolated fixed points.

In the present article we obtain the full classification of complex cobordism classes $\sigma \in \Omega_U$ containing a manifold with simple action of \mathbb{Z}/p . The description is given

both in terms of the coefficients of universal formal group law of “geometric cobordisms” (theorem 3.1) and in terms of the characteristic numbers (theorem 4.2, corollary 4.3). Our corollary 4.3 can be also viewed as a cohomological obstruction to the existence of a simple \mathbb{Z}/p -action (or an action with isolated fixed points) on a manifold.

The classification problem for strictly simple actions of \mathbb{Z}/p was completely solved by Conner and Floyd in [CF1]. (A strictly simple action from definition 1.1 was called in [CF1] just “an action of \mathbb{Z}/p with fixed point set having the trivial normal bundle”.) Note, that even in the special case of an action with finite number of isolated fixed points the notions of simple and strictly simple action differ (see examples below). The Conner–Floyd results follow from the results of our paper. At the same time, it seems to us that the approach used in [CF1] does not allow to obtain our more general result.

The applications of the formal group law theory to \mathbb{Z}/p -actions were firstly discussed in the pioneer article [N1]. The formal group law theory itself arises in topology due to so-called formal group law of geometric cobordisms. The problem solved in our article was firstly formulated in [BN]. There was obtained the formula expressing the mod p cobordism class of a manifold M with a simple action of \mathbb{Z}/p in terms of the cobordism classes of fixed submanifolds and the weights of the action (see formula (9)). Actually, the first results on the problem were obtained even earlier, in [K]. Particularly, there was proved the statement mentioned in our article as corollary 4.4. In [K], as well as in our work, the set of cobordism classes of manifolds with a simple \mathbb{Z}/p -action is handled as the Ω_U -module spanned by certain coefficients of the power system determined by the formal group law of geometric cobordisms. (Here Ω_U is the complex cobordism ring of point, which is isomorphic to the polynomial ring $\mathbb{Z}[a_1, a_2, \dots]$, $\deg a_i = -2i$, as it was shown by Milnor and Novikov.) However, our new choice of generators for the above Ω_U -module allowed us to present the answer for the classification problem in terms of the characteristic numbers, which of course is much more explicit.

2. THE RING $U^*(\mathbb{Z}/p)$ OF EQUIVARIANT COBORDISMS WITH FREE \mathbb{Z}/p -ACTION AND THE CONNER–FLOYD EQUATIONS

Now, let us start with a stably complex manifold M with a simple action of \mathbb{Z}/p . Let the fixed submanifolds represent the cobordism classes $\lambda_j \in \Omega_U$ and have weights $(x_1^{(j)}, \dots, x_{m_j}^{(j)})$ in their (trivial) normal bundles. Here $2m_j + \dim \lambda_j = \dim M$, $(x_k^{(j)}) \in (\mathbb{Z}/p)^*$, $k = 1, \dots, m_j$. These data define the cobordism class of M in Ω_U up to elements from $p\Omega_U$ (cf. [BN]). This follows from the fact that the cobordism class of a manifold with free \mathbb{Z}/p -action (i.e. without fixed points) belongs to $p\Omega_U$, and vice versa, any cobordism class from $p\Omega_U$ can be represented by a manifold with free action of \mathbb{Z}/p .

Let $B\mathbb{Z}/p$ denote the classifying space for the group \mathbb{Z}/p (the infinite-dimensional lens space). To each fixed submanifold of the cobordism class $\lambda_j \in \Omega_U$ with weights $(x_1^{(j)}, \dots, x_{m_j}^{(j)})$, $2m_j + \dim \lambda_j = \dim M$, one can associate an element

$$\alpha_{2m_j-1}(x_1^{(j)}, \dots, x_{m_j}^{(j)}) \in U_{2m_j-1}(B\mathbb{Z}/p)$$

called the Conner–Floyd invariant (cf. [N1]). To define it we mention that the bordism group $U_*(B\mathbb{Z}/p)$ is isomorphic to the group of free \mathbb{Z}/p -equivariant bordisms (cf. [CF1]). This isomorphism takes an equivariant bordism class represented by a manifold N with free action of \mathbb{Z}/p to the bordism class in $U_*(B\mathbb{Z}/p)$ determined by the classifying map $N/(\mathbb{Z}/p) \rightarrow B\mathbb{Z}/p$. Then $\alpha_{2m_j-1}(x_1^{(j)}, \dots, x_{m_j}^{(j)})$ is defined as the equivariant bordism class of the unit sphere in the fibre of (trivial) normal

bundle to λ_j . If we write this unit sphere as

$$\{(z_1, \dots, z_{m_j}) \in \mathbb{C}^{m_j} : |z_1|^2 + \dots + |z_{m_j}|^2 = 1\},$$

then the generator of \mathbb{Z}/p acts on it as

$$(z_1, \dots, z_{m_j}) \rightarrow (e^{2\pi i x_1^{(j)}/p} z_1, \dots, e^{2\pi i x_{m_j}^{(j)}/p} z_{m_j}).$$

The complex cobordism ring of $\mathbb{C}P^\infty$ is $\Omega_U[[v]]$, where $v = c_1^U(\zeta)$ is the first cobordism Chern class of the universal line bundle ζ over $\mathbb{C}P^\infty$ (this follows in the standard way from the Atiyah–Hirzebruch spectral sequence in cobordisms). There is a canonical bundle $f : B\mathbb{Z}/p \rightarrow \mathbb{C}P^\infty$ with fibre S^1 . The complex cobordism ring of $B\mathbb{Z}/p$ is $U^*(B\mathbb{Z}/p) = \Omega_U[[u]]/([u]_p = 0)$ (cf. [BN]), where $u = f^*(v)$ and $[u]_p = pu + \dots$ is the p -th power in the (universal) formal group law of geometric cobordisms (a series with coefficients in Ω_U). We have $D(\alpha_i(1, \dots, 1)) = u^{n-i}$, where D is the Poincaré–Atiyah duality operator from $U_*(L_p^{2n-1})$ to $U^*(L_p^{2n-1})$ (here L_p^{2n-1} is the standard $(2n-1)$ -dimensional lens space). It follows that the Ω_U -module $\tilde{U}_*(B\mathbb{Z}/p)$ is generated by the elements $\alpha_{2i-1}(1, \dots, 1)$ with the following relations:

$$(1) \quad 0 = \frac{[u]_p}{u} \cap \alpha_{2i-1}(1, \dots, 1).$$

Here \cap denotes the cobordism \cap -product, and

$$u^k \cap \alpha_{2i-1}(1, \dots, 1) = \alpha_{2(i-k)-1}(1, \dots, 1).$$

It was shown in [N2] that

$$(2) \quad \alpha_{2k-1}(x_1, \dots, x_k) = \left(\prod_{j=1}^k \frac{u}{[u]_{x_j}} \right) \cap \alpha_{2k-1}(1, \dots, 1).$$

Here and below $[u]_k$ denotes the k -th power in the formal group law of geometric cobordisms. Adding the elements $\alpha_{2k-1}(x_1, \dots, x_k)$, $x_j \neq 1 \pmod p$, to the set of generators for the module $\tilde{U}_*(B\mathbb{Z}/p)$ and adding relations (2) to relations (1) we obtain the $\Omega_U \otimes \mathbb{Z}_{(p)}$ -free resolution of the module $\tilde{U}_*(B\mathbb{Z}/p)$ (here $\mathbb{Z}_{(p)}$ is a ring of rational numbers whose denominators are relatively prime with p , i.e. the ring of integer p -adics):

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \tilde{U}_*(B\mathbb{Z}/p) \longrightarrow 0.$$

Here F_0 is the free $\Omega_U \otimes \mathbb{Z}_{(p)}$ -module spanned by $\alpha_{2k-1}(x_1, \dots, x_k)$, and F_1 is the free $\Omega_U \otimes \mathbb{Z}_{(p)}$ -module spanned by the elements

$$a(x_1, \dots, x_k) = \alpha_{2k-1}(x_1, \dots, x_k) - \left(\prod_{j=1}^k \frac{u}{[u]_{x_j}} \right) \cap \alpha_{2k-1}(1, \dots, 1),$$

$$a_k = \frac{[u]_p}{u} \cap \alpha_{2k-1}(1, \dots, 1),$$

(see formulae (1) and (2)).

Hence, a simple action of \mathbb{Z}/p on M gives rise to a certain relation between the elements $\alpha_{2k-1}(x_1, \dots, x_k)$ in $\tilde{U}_*(B\mathbb{Z}/p)$. Since any element from $\tilde{U}_*(B\mathbb{Z}/p)$ corresponds to a bordism class of manifold with free \mathbb{Z}/p -action, the converse is also true: any relation in $\tilde{U}_*(B\mathbb{Z}/p)$ of the form

$$\sum_j \lambda_j \alpha_{2m_j-1}(x_1^{(j)}, \dots, x_{m_j}^{(j)}) = 0, \quad \lambda_j \in \Omega_U, \quad 2m_j + \dim \lambda_j = \dim M,$$

is realized on a certain manifold M with a simple action of \mathbb{Z}/p whose cobordism class in Ω_U is uniquely determined up to elements from $p\Omega_U$. This manifold M can be constructed as follows. The relation in $\tilde{U}_*(B\mathbb{Z}/p)$ gives us a manifold with free

\mathbb{Z}/p -action whose boundary is the union of manifolds of the form $\lambda_j \times S^{2m_j-1}$. Then we glue “covers” of the form $\lambda_j \times D^{2m_j}$ to the boundary to get the closed manifold M , which realizes the above relation. Therefore, one can define the “realization homomorphism”

$$(3) \quad \Phi : F_1 \rightarrow \Omega_U/p\Omega_U = \Omega_U \otimes \mathbb{Z}/p.$$

It takes a relation between the elements $\alpha_{2k-1}(x_1, \dots, x_k) \in \tilde{U}_*(B\mathbb{Z}/p)$ to the mod p cobordism class of the manifold that realizes this relation as described above. The Conner–Floyd results (cf. [CF1], see also [K]) give us the following values of Φ on the basis relations from F_1 :

$$\begin{aligned} \Phi(a(x_1, \dots, x_k)) &= \left\langle \prod_{i=1}^k \frac{u}{[u]_{x_i}} \right\rangle_k \mod p \in \Omega_U/p\Omega_U, \\ \Phi(a_k) &= -\left\langle \frac{[u]_p}{u} \right\rangle_k \mod p \in \Omega_U/p\Omega_U, \end{aligned}$$

where $\langle \rangle_k$ stands for the coefficient of u^k . Following [BN], let us consider the coefficient ring $\Lambda(1) \subset \Omega_U$ of the power system $[u]_k$ (i.e. the subring of Ω_U generated by the coefficients of the series $[u]_k$ for all $k > 0$), and its positive part $\Lambda^+(1)$, consisting of elements of non-zero degree in Ω_U . It follows that $\text{Im } \Phi = \tilde{\Lambda}(1) \otimes \mathbb{Z}/p$, where $\tilde{\Lambda}(1) = \Lambda^+(1) \cdot \Omega_U$ is the Ω_U -module spanned by $\Lambda^+(1)$. The homomorphism Φ lifts to the homomorphism $F_1 \rightarrow \tilde{\Lambda}(1) \otimes \mathbb{Z}_{(p)}$, or to the homomorphism $F_1 \rightarrow \tilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$, where $\tilde{\Lambda}_p(1)$ is the Ω_U -module spanned by $\Lambda^+(1)$ and p . Both these homomorphisms will be denoted by the same letter Φ .

Thus, the problem of description of cobordism classes of manifolds with a simple \mathbb{Z}/p -action is equivalent to the problem of description of $\Omega_U \otimes \mathbb{Z}/p$ -module $\tilde{\Lambda}(1) \otimes \mathbb{Z}/p$, or $\Omega_U \otimes \mathbb{Z}_{(p)}$ -modules $\tilde{\Lambda}(1) \otimes \mathbb{Z}_{(p)}$, $\tilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$. These modules are ideals in $\Omega_U \otimes \mathbb{Z}/p$ and $\Omega_U \otimes \mathbb{Z}_{(p)}$ correspondingly.

3. THE GENERATOR SETS FOR $\Omega_U \otimes \mathbb{Z}/p$ -MODULE $\tilde{\Lambda}(1) \otimes \mathbb{Z}/p$ AND $\Omega_U \otimes \mathbb{Z}_p$ -MODULES $\tilde{\Lambda}(1) \otimes \mathbb{Z}_p$, $\tilde{\Lambda}_p(1) \otimes \mathbb{Z}_p$.

Let us write

$$[u]_k = ku + \sum_{n \geq 1} \alpha_n^{(k)} u^{n+1},$$

i.e. $\alpha_n^{(k)} \in \Omega_U^{-2n}$ are the coefficients of the power system. The module $\tilde{\Lambda}(1) \otimes \mathbb{Z}_{(p)}$ is therefore generated by $\alpha_n^{(k)}$, $k > 1$, $n \geq 1$, over $\Omega_U \otimes \mathbb{Z}_{(p)}$. The following theorem shows that the generator set can be taken in such a way that there is only one generator in each dimension.

Theorem 3.1. *One can take the following coefficients $\alpha_n \in \Omega_U^{-2n}$ as generators of the $\Omega_U \otimes \mathbb{Z}_{(p)}$ -module $\tilde{\Lambda}(1) \otimes \mathbb{Z}_{(p)}$:*

$$(4) \quad \alpha_n = \begin{cases} \alpha_n^{(p_1)}, & \text{if } n \text{ is not divisible by } p-1, \\ \alpha_n^{(p)}, & \text{if } n \text{ is divisible by } p-1, \end{cases} \quad n = 1, 2, \dots,$$

Here p_1 is any prime generator of the cyclic group $(\mathbb{Z}/p)^*$.

Remark. It follows from the Dirichlet theorem that one can choose a *prime* generator p_1 of the cyclic group $(\mathbb{Z}/p)^*$.

Proof of theorem 3.1. First, let us consider the coefficients $\alpha_n^{(r)}$ for non-prime r . So, let $r = p_1 q$ with prime p_1 . Since $[x]_r = [[x]_{p_1}]_q$, we have:

$$\begin{aligned} rx + \sum_n \alpha_n^{(r)} x^{n+1} &= q[x]_{p_1} + \sum_n \alpha_n^{(q)} ([x]_{p_1})^{n+1} \\ &= p_1 qx + q \sum_n \alpha_n^{(p_1)} x^{n+1} + \sum_n \alpha_n^{(q)} \left(p_1 x + \sum_m \alpha_m^{(p_1)} x^{m+1} \right)^{n+1}. \end{aligned}$$

Taking the coefficient of x^{r+1} in both sides of the above identity, we get

$$\alpha_n^{(r)} = P(\alpha_1^{(p_1)}, \dots, \alpha_n^{(p_1)}, \alpha_1^{(q)}, \dots, \alpha_n^{(q)}),$$

where P is a polynomial with integer coefficients (and zero constant term). Hence, we can write

$$\alpha_n^{(r)} = \lambda_1 \alpha_1^{(p_1)} + \dots + \lambda_n \alpha_n^{(p_1)} + \mu_1 \alpha_1^{(q)} + \dots + \mu_n \alpha_n^{(q)}, \quad \lambda_i, \mu_i \in \Omega_U.$$

Therefore, the coefficients $\alpha_n^{(r)}$, $r = p_1 q$ can be excluded from the set of generators for the $\Omega_U \otimes \mathbb{Z}_{(p)}$ -module $\tilde{\Lambda}(1) \otimes \mathbb{Z}_{(p)}$. Now, if q is still not prime, we repeat the above procedure until we arrive at a set of generators consisting only of coefficients $\alpha_n^{(p_1)}$ with prime p_1 . Now, what we need to show is that this set of generators can still be reduced to the set (4).

Note, that for any (prime) generator p_1 of the cyclic group $(\mathbb{Z}/p)^*$ one can take the coefficient $\alpha_1^{(p_1)}$ as a generator of $\tilde{\Lambda}(1) \otimes \mathbb{Z}_{(p)}$ in the dimension -2 (i.e. in Ω_U^{-2}). Indeed, let p_2 be any prime. Then $[[x]_{p_2}]_{p_1} = [[x]_{p_1}]_{p_2}$. Hence,

$$\begin{aligned} (5) \quad p_1 p_2 x + p_1 \sum_n \alpha_n^{(p_2)} x^{n+1} + \sum_n \alpha_n^{(p_1)} \left(p_2 x + \sum_m \alpha_m^{(p_2)} x^{m+1} \right)^{n+1} \\ = p_2 p_1 x + p_2 \sum_n \alpha_n^{(p_1)} x^{n+1} + \sum_n \alpha_n^{(p_2)} \left(p_1 x + \sum_m \alpha_m^{(p_1)} x^{m+1} \right)^{n+1}. \end{aligned}$$

Taking the coefficient of x^2 in both sides of the above identity, we get $p_1 \alpha_1^{(p_2)} + p_2^2 \alpha_1^{(p_1)} = p_2 \alpha_1^{(p_1)} + p_1^2 \alpha_1^{(p_2)}$. Hence, $(p_1 - p_1^2) \alpha_1^{(p_2)} = (p_2 - p_2^2) \alpha_1^{(p_1)}$. Since p_1 is a generator of $(\mathbb{Z}/p)^*$, the element $p_1 - p_1^2$ is invertible in $\mathbb{Z}_{(p)}$. So, $\alpha_1^{(p_2)} = \lambda \alpha_1^{(p_1)}$ with $\lambda \in \mathbb{Z}_{(p)} \subset \Omega_U \otimes \mathbb{Z}_{(p)}$. Thus, for any prime $p_2 \neq p_1$ the coefficient $\alpha_1^{(p_2)}$ is a multiple of $\alpha_1^{(p_1)}$, and that is why it can be excluded from the set of generators for $\tilde{\Lambda}(1) \otimes \mathbb{Z}_{(p)}$.

Now, consider the coefficient system $\alpha_1, \dots, \alpha_k, \dots$ introduced in the theorem. (That is, α_i is the coefficient of x^{i+1} in the series $[x]_{p_1}$ if i is not divisible by $p-1$, and is the coefficient of x^{i+1} in the series $[x]_p$ if i is divisible by $p-1$.) By induction, we may suppose that this coefficient system is a set of generators for $\tilde{\Lambda}(1) \otimes \mathbb{Z}_{(p)}$ in all dimensions up to $-2(n-1)$. Hence, for any q and $k \leq n-1$ one has

$$(6) \quad \alpha_k^{(q)} = \lambda_1^{(q)} \alpha_1 + \dots + \lambda_k^{(q)} \alpha_k$$

with $\lambda_i^{(q)} \in \Omega_U \otimes \mathbb{Z}_{(p)}$. We are going to prove that $\alpha_n^{(q)}$ can be also decomposed in such a way. It follows from the above argument that we can consider only prime q .

First, suppose that n is not divisible by $p-1$. Hence, $\alpha_n = \alpha_n^{(p_1)}$, where p_1 is a generator of $(\mathbb{Z}/p)^*$. Let p_2 be any prime. Taking the coefficient of x^{n+1} in both sides of (5), we obtain

$$\begin{aligned} p_1 \alpha_n^{(p_2)} + p_2^{n+1} \alpha_n^{(p_1)} + \mu_1 \alpha_1 + \dots + \mu_{n-1} \alpha_{n-1} \\ = p_2 \alpha_n^{(p_1)} + p_1^{n+1} \alpha_n^{(p_2)} + \nu_1 \alpha_1 + \dots + \nu_{n-1} \alpha_{n-1} \end{aligned}$$

Here we expressed coefficients $\alpha_k^{(p_1)}$, $\alpha_k^{(p_2)}$, $k < n$, as a linear combinations of generators $\alpha_1, \dots, \alpha_{n-1}$, i.e. $\mu_i, \nu_i \in \Omega_U \otimes \mathbb{Z}_{(p)}$. Therefore,

$$(7) \quad p_1(1 - p_1^n)\alpha_n^{(p_2)} = (p_2 - p_2^{n+1})\alpha_n^{(p_1)} + (\nu_1 - \mu_1)\alpha_1 + \dots + (\nu_{n-1} - \mu_{n-1})\alpha_{n-1}.$$

Since p_1 is a generator of $(\mathbb{Z}/p)^*$ and n is not divisible by $p - 1$, we deduce that $p_1(1 - p_1^n)$ is invertible in $\mathbb{Z}_{(p)}$. Thus, it follows from (7) that $\alpha_n^{(p_2)}$ is a linear combination of $\alpha_1, \dots, \alpha_{n-1}$ and $\alpha_n = \alpha_n^{(p_1)}$ with coefficients from $\Omega_U \otimes \mathbb{Z}_{(p)}$.

Now, suppose that n is divisible by $p - 1$, i.e. $\alpha_n = \alpha_n^{(p)}$. Before we proceed further, let us make some preliminary remarks. It is well known (Milnor, Novikov), that the complex cobordism coefficient ring Ω_U is a polynomial ring: $\Omega_U = \mathbb{Z}[a_1, a_2, \dots, a_n, \dots]$, $a_n \in \Omega_U^{-2n}$. The ring Ω_U is the coefficient ring of the (universal) formal group law of geometric cobordisms (cf. [Q], [BN]). This formal group law has a logarithm series with coefficients in $\Omega_U \otimes \mathbb{Q}$, namely $g(u) = u + \sum_n \frac{\mathbb{C}P^n}{n+1} u^{n+1}$, cf. [N1]. Hence, the coefficient ring of the logarithm is $\Omega_U(\mathbb{Z}) := \mathbb{Z}[b_1, b_2, \dots, b_n, \dots]$, where $b_n = \frac{\mathbb{C}P^n}{n+1}$. It is well known that this ring is the maximal subring of $\Omega_U \otimes \mathbb{Q}$ on which all cohomological characteristic numbers take integer values. One can choose generator sets $\{a_i^*\}$, $\{b_i^*\}$ for the rings Ω_U , $\Omega_U(\mathbb{Z})$ such that the inclusion $\iota_0 : \Omega_U \rightarrow \Omega_U(\mathbb{Z})$ is as follows:

$$\iota_0(a_i^*) = \begin{cases} p \cdot b_i^*, & \text{if } i = p^k - 1 \text{ for some } k > 0, \\ b_i^* & \text{otherwise.} \end{cases}$$

Let B^+ be the set of elements of degree $\neq 0$ in the ring $B := \Omega_U(\mathbb{Z})$. Then $(B^+)^2$ consists of elements in $\Omega_U(\mathbb{Z})$ that are decomposable into the product of two non-trivial factors. The map $\iota_0 : \Omega_U \rightarrow \Omega_U(\mathbb{Z})$ takes the coefficients $\alpha_n^{(p)}$ of the series $[x]_p$ to the element of the form $(p - p^{n+1})b_n + ((B^+)^2)$ (cf. [B2, p. 22]). Therefore, the coefficients $\alpha_{p^k-1}^{(p)}$ can be taken as multiplicative generators of $\Omega_U \otimes \mathbb{Z}_{(p)}$ in dimensions $p^k - 1$. In other dimensions $l \neq p^k - 1$ we have $\alpha_l^{(p)} \in p\Omega_U$, i.e. $\alpha_l^{(p)}$ is divisible by p in Ω_U .

Now, let us return to the proof of theorem 3.1. Let us rewrite the identity (5) substituting p for p_1 :

$$\begin{aligned} & pp_2x + p \sum_m \alpha_m^{(p_2)} x^{m+1} + \sum_m \alpha_m^{(p)} \left(p_2x + \alpha_1^{(p_2)} x^2 + \alpha_2^{(p_2)} x^3 + \dots \right)^{m+1} \\ &= p_2px + p_2 \sum_m \alpha_m^{(p)} x^{m+1} + \sum_m \alpha_m^{(p_2)} \left(px + \alpha_1^{(p)} x^2 + \alpha_2^{(p)} x^3 + \dots \right)^{m+1}. \end{aligned}$$

Taking the coefficient of x^{n+1} in both sides, we get

$$\begin{aligned} & p\alpha_n^{(p_2)} + p_2^{n+1}\alpha_n^{(p)} + \left\langle \sum_{m < n} \alpha_m^{(p)} (p_2x + \alpha_1^{(p_2)} x^2 + \alpha_2^{(p_2)} x^3 + \dots)^{m+1} \right\rangle_{n+1} \\ &= p_2\alpha_n^{(p)} + p^{n+1}\alpha_n^{(p_2)} + \left\langle \sum_{m < n} \alpha_m^{(p_2)} (px + \alpha_1^{(p)} x^2 + \alpha_2^{(p)} x^3 + \dots)^{m+1} \right\rangle_{n+1}, \end{aligned}$$

where $\langle \cdot \rangle_{n+1}$ denotes the coefficient of x^{n+1} . Let us write again the coefficients $\alpha_m^{(p_2)}$ for $m < n$ as linear combinations of generators $\alpha_1, \dots, \alpha_m$. Since $\alpha_m^{(p)} \in p\Omega_U$ for

$m \neq p^k - 1$, the last identity can be rewritten as

$$(8) \quad p(1 - p^n)\alpha_n^{(p_2)} = p_2(1 - p_2^n)\alpha_n^{(p)} + p(\mu_1\alpha_1 + \dots + \mu_{n-1}\alpha_{n-1}) \\ - \left\langle \sum_{k: p^k - 1 < n} \alpha_{p^k - 1}^{(p)} \left(p_2x + \alpha_1^{(p_2)}x^2 + \alpha_2^{(p_2)}x^3 + \dots \right)^{p^k} \right\rangle_{n+1} \\ + \left\langle \sum_{m < n} \alpha_m^{(p_2)} \left(\alpha_{p-1}^{(p)}x^p + \alpha_{p^2-1}^{(p)}x^{p^2} + \dots + \alpha_{p^k-1}^{(p)}x^{p^k} + \dots \right)^{m+1} \right\rangle_{n+1}$$

for some $\mu_i \in \Omega_U \otimes \mathbb{Z}_{(p)}$. The last two summands in the above formula can be rewritten as $\alpha_{p-1}^{(p)}\nu_1 + \alpha_{p^2-1}^{(p)}\nu_2 + \dots + \alpha_{p^k-1}^{(p)}\nu_k$, where $k = [\log_p(n+1)]$, $\nu_i \in \Omega_U$. The coefficients $\alpha_{p^i-1}^{(p)}$ are multiplicative generators of $\Omega_U \otimes \mathbb{Z}_{(p)}$ in the dimensions $p^i - 1$. Since $\Omega_U \otimes \mathbb{Z}_{(p)}$ is a polynomial ring, one has $\nu_i \in p\Omega_U \otimes \mathbb{Z}_{(p)}$, i.e. ν_i is divisible by p in $\Omega_U \otimes \mathbb{Z}_{(p)}$. Let $\nu_i = p\kappa_i$ with $\kappa_i \in \Omega_U \otimes \mathbb{Z}_{(p)}$. Then (8) gives

$$p(1 - p^n)\alpha_n^{(p_2)} = p_2(1 - p_2^n)\alpha_n^{(p)} + p(\mu_1\alpha_1 + \dots + \mu_{n-1}\alpha_{n-1}) \\ + p(\alpha_{p-1}^{(p)}\kappa_1 + \alpha_{p^2-1}^{(p)}\kappa_2 + \dots + \alpha_{p^k-1}^{(p)}\kappa_k),$$

where $k = [\log_p(n+1)]$, $\mu_i, \kappa_i \in \Omega_U \otimes \mathbb{Z}_{(p)}$. Since n is divisible by $p-1$, it follows that $1 - p_2^n$ is divisible by p (for $p_2 \neq p$). Hence, the whole above identity is divisible by p . Dividing it by p and mentioning that $1 - p^n$ is invertible in $\Omega_U \otimes \mathbb{Z}_{(p)}$, we obtain that $\alpha_n^{(p_2)}$ is decomposable as

$$\alpha_n^{(p_2)} = \frac{p_2(1 - p_2^n)}{p(1 - p^n)}\alpha_n^{(p)} + \lambda_1\alpha_1 + \dots + \lambda_{n-1}\alpha_{n-1}$$

with $\lambda_i \in \Omega_U \otimes \mathbb{Z}_{(p)}$. Thus, setting

$$\lambda_n = \frac{p_2(1 - p_2^n)}{p(1 - p^n)} \in \Omega_U \otimes \mathbb{Z}_{(p)},$$

we get a decomposition of type (6) for $\alpha_n^{(p_2)}$ (note that $\alpha_n = \alpha_n^{(p)}$), which completes the proof of theorem 3.1. \square

Corollary 3.2. *Let p_1 be a prime generator of the cyclic group $(\mathbb{Z}/p)^*$. There is the following set of generators for the $\Omega_U \otimes \mathbb{Z}_{(p)}$ -module $\tilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$:*

$$\alpha_n = \begin{cases} p, & \text{if } n = 0, \\ \alpha_n^{(p_1)}, & \text{if } n \text{ is not divisible by } p-1, \\ \alpha_{p^k-1}^{(p)}, & \text{if } n = p^k - 1, k = 1, 2, \dots \end{cases}$$

The $\Omega_U \otimes \mathbb{Z}/p$ -module $\tilde{\Lambda}(1) \otimes \mathbb{Z}/p$ has the following generator set:

$$\alpha_n = \begin{cases} \alpha_n^{(p_1)}, & \text{if } n \text{ is not divisible by } p-1, \\ \alpha_{p^k-1}^{(p)}, & \text{if } n = p^k - 1, k = 1, 2, \dots \end{cases}$$

Remark. In both cases there no generators in the dimensions n divisible by $p-1$ other than $p^k - 1$.

Proof. Consider the set of generators for $\tilde{\Lambda}(1) \otimes \mathbb{Z}_{(p)}$ constructed in theorem 3.1. If n is divisible by $p-1$ and $n \neq p^k - 1$, the elements α_n are divisible by p , i.e. lie in $p\Omega_U$. All other α_n do not belong to $p\Omega_U$. \square

4. COHOMOLOGICAL DESCRIPTION OF THE SET OF COBORDISM CLASSES OF MANIFOLDS WITH A SIMPLE ACTION OF \mathbb{Z}/p AND SOME COROLLARIES

In this section we use the previously obtained description of the $\Omega_U \otimes \mathbb{Z}_{(p)}$ -module $\tilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$ to prove the result analogous to the well-known Stong-Hattori theorem [CF2]. Namely, we will describe the set of cobordism classes of manifolds with a simple \mathbb{Z}/p -action in terms of the characteristic numbers.

As it was shown in [BN], the homomorphism $\Phi : F_1 \rightarrow \tilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$ (see (3) and following discussion) can be extended to a homomorphism $\gamma_p : F_0 \rightarrow \Omega_U(\mathbb{Z}) \otimes \mathbb{Z}_{(p)}$ such that

$$\gamma_p(\alpha_{2k-1}(x_1, x_2, \dots, x_k)) = \left\langle \left(\prod_{j=1}^k \frac{u}{[u]_{x_j}} \right) \frac{pu}{[u]_p} \right\rangle_k,$$

where $\alpha_{2k-1}(x_1, x_2, \dots, x_k) \in F_0$ is the Conner–Floyd invariant (see (2)). In particular,

$$\gamma_p(\alpha_{2k-1}(1, \dots, 1)) = \left\langle \frac{pu}{[u]_p} \right\rangle_k.$$

Hence, for any simple action of \mathbb{Z}/p on M^{2n} the $\text{mod } p$ cobordism class of M^{2n} can be expressed in terms of the cobordism classes $\lambda_j \in \Omega_U$ of fixed submanifolds and the weights $x_k^{(j)} \in (\mathbb{Z}/p)^*$ in the corresponding (trivial) normal bundles as follows:

$$(9) \quad [M^{2n}] \equiv \sum_j \lambda_j \gamma_p(x_1^{(j)}, \dots, x_{m_j}^{(j)}) \text{ mod } p\Omega_U.$$

Now, the following question arises: which elements of the form

$$\sum_j \lambda_j \gamma_p(x_1^{(j)}, \dots, x_{m_j}^{(j)}) \in \Omega_U(\mathbb{Z}) \otimes \mathbb{Z}_{(p)}$$

are cobordism classes of manifolds with a simple \mathbb{Z}/p -action? This question was firstly posed in [BN] and is analogous to the Milnor–Hirzebruch problem of describing the set of elements in $\Omega_U(\mathbb{Z})$ that are cobordism classes of (stably complex) manifolds. While the Milnor–Hirzebruch problem is solved by the Stong–Hattori theorem, the answer to the above question is given in our theorem 4.2. We will need the following definition.

Definition 4.1. Let $\omega = \sum_{i=1}^l k_i \cdot (i)$, $i, k_i \in \mathbb{Z}$, $i > 0$, $k_i \geq 0$, be a partition of $n = \|\omega\| = \sum_i k_i \cdot i$ (i.e. n is decomposed into the sum of positive integers, and the number i enters this sum k_i times). We say that the partition ω is *divisible by $p-1$* , if all i such that $k_i \neq 0$ are divisible by $p-1$ (i.e. all the summands are divisible by $p-1$; obviously, such partitions exist only for those n divisible by $p-1$). We say that the partition ω is *non p -adic*, if for any $j > 0$ one has $k_{p^j-1} = 0$ (i.e. there no summands of the form $p^j - 1$).

For each partition $\omega = \sum_{i=1}^l k_i(i)$ let us put $\|\omega\| = \sum_i k_i \cdot i$ and $|\omega| = \sum_i k_i$ (i.e. ω is a partition of $\|\omega\|$ with number of summands equals $|\omega|$). A partition ω defines a characteristic class s_ω as follows. Let us consider the smallest symmetric polynomial in $x_1, \dots, x_{|\omega|}$ containing the monomial

$$(x_1 \cdots x_{k_1})(x_{k_1+1}^2 \cdots x_{k_1+k_2}^2) \cdots (x_{|\omega|-k_{l-1}+1}^l \cdots x_{|\omega|}^l).$$

This polynomial defines a characteristic class in a usual way (cf. [S]); in order to express it in terms of the Chern characteristic classes one should write it as a polynomial in the elementary symmetric functions σ_i , and then substitute c_i for σ_i . Given an $2n$ -dimensional stably complex manifold M^{2n} , one can define the cohomological characteristic numbers $s_\omega(M^{2n}) := s_\omega(TM^{2n})[M^{2n}] \in \mathbb{Z}$ for all partitions ω such that $\|\omega\| = n$. The K -theory characteristic numbers $s_\omega(M^{2n}) \in \mathbb{Z}$ (cf. [CF2]) are

defined for all partitions ω such that $\|\omega\| \leq n$; they coincide with the cohomological numbers for $\|\omega\| = n$, while for $\omega = 0$ the corresponding K -theory characteristic number is the Todd genus.

Theorem 4.2. *An element $\sigma \in \Omega_U(\mathbb{Z})^{-2n} \otimes \mathbb{Z}_{(p)}$ belongs to the $\Omega_U \otimes \mathbb{Z}_{(p)}$ -module $\tilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$ and therefore, is the cobordism class of a manifold with a simple \mathbb{Z}/p -action, if and only if all its K -theory characteristic numbers $s_\omega(\sigma)$, $\omega = \sum_i k_i \cdot (i)$, $\|\omega\| = \sum_i k_i \cdot i \leq n$, lie in $\mathbb{Z}_{(p)}$, and for all partitions ω divisible by $p-1$ the cohomological characteristic numbers $s_\omega(\sigma)$, $\|\omega\| = n$, are zero modulo p .*

Proof. (a) Necessity.

Let $\sigma \in \tilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$. Note that the set of generators for the $\Omega_U \otimes \mathbb{Z}_{(p)}$ -module $\tilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$ described in corollary 3.2 has the following property: each of its elements $\alpha_i \in \tilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$ is a multiplicative generator of $\Omega_U \otimes \mathbb{Z}_{(p)}$ in dimension $-2i$. However, this set of generators for $\tilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$ has no elements in dimensions $-2i$ such that i is divisible by $p-1$ and $i \neq p^k - 1$. So, we add any generators α_i in these missing dimensions to get the whole set of multiplicative generators for $\Omega_U \otimes \mathbb{Z}_{(p)}$. Now we have $\Omega_U \otimes \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}[\alpha_1, \alpha_2, \dots]$.

Since $\sigma \in \tilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)} \subset \Omega_U \otimes \mathbb{Z}_{(p)}$, it follows from the Stong–Hattori theorem that all the K -characteristic numbers $s_\omega(\sigma)$, $\|\omega\| \leq n$, lie in $\mathbb{Z}_{(p)}$.

If n is not divisible by $p-1$, then there are no partitions ω divisible by $p-1$.

Now, let $n = m(p-1)$. One can write σ as a homogeneous polynomial of degree $-2m(p-1)$ in α_i :

$$(10) \quad \sigma = \sum_{\|\omega\|=m(p-1)} r_\omega \alpha_\omega = r_{m(p-1)} \alpha_{m(p-1)} + \dots,$$

where $\alpha_\omega = \alpha_1^{k_1} \cdot \alpha_2^{k_2} \cdots \alpha_l^{k_l}$ for $\omega = \sum_i k_i \cdot (i)$. It follows from the description of $\tilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$ given in corollary 3.2 that $\sigma \in \tilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$ if and only if the coefficients r_ω in the decomposition (10) are zero modulo p for all non p -adic and divisible by $p-1$ partitions ω .

Consider the Chern–Dold character $\text{ch}_U : U^*(\cdot) \rightarrow H^*(\cdot; \Omega_U \otimes \mathbb{Q})$ in cobordisms [B1]:

$$\text{ch}_U(v) = t + \sum_{i \geq 1} \beta_i t^{i+1}.$$

Here $v = c_1^U(\zeta) \in U^2(\mathbb{C}P^\infty)$ is the first cobordism Chern class of the universal line bundle, $t = c_1^H(\zeta) \in H^2(\mathbb{C}P^\infty)$ is the same Chern class in cohomologies, and the coefficients β_i are from $\Omega_U(\mathbb{Z})$. Then for any $\sigma \in \Omega_U^{-2n}$ holds

$$(11) \quad \sigma = \sum_{\|\omega\|=n} s_\omega(\sigma) \beta_\omega,$$

where $\beta_\omega = \beta_1^{k_1} \cdot \beta_2^{k_2} \cdots \beta_l^{k_l}$ for $\omega = \sum_i k_i \cdot (i)$. The coefficient ring $\mathbb{Z}[\beta_1, \beta_2, \dots]$ of the Chern–Dold character coincides with $B = \Omega_U(\mathbb{Z})$ (cf. [B1]). Hence,

$$\alpha_i = \begin{cases} e_i \cdot \beta_i + ((B^+)^2) & \text{if } i \neq p^k - 1, \\ pe_i \cdot \beta_i + p((B^+)^2) & \text{if } i = p^k - 1 \end{cases}$$

with invertible $e_i \in \mathbb{Z}_{(p)}$. Now, let us write σ as a homogeneous polynomial in β_i . Since all $\beta_i \in \Omega_U(\mathbb{Z})$ have integer cohomological characteristic numbers, to prove the necessity of the theorem it suffices to show that the coefficient of β_ω in the decomposition of σ is zero modulo p if the partition $\omega = \sum_i k_i \cdot (i)$ is divisible by $p-1$. This coefficient is the homological characteristic number $s_\omega(\sigma)$ (see (11)),

which can be decomposed as follows (see (10)):

$$(12) \quad s_\omega(\sigma) = \sum_{\omega': \omega' \supset \omega} r_{\omega'} s_\omega(\alpha_{\omega'}),$$

where $\omega' \supset \omega$ means that ω refines ω' . This coefficient is divisible by p . Indeed, if the partition $\omega' = \sum_i k'_i \cdot (i)$ is divisible by $p-1$ and non p -adic, then $r_{\omega'}$ is zero modulo p , since $\sigma \in \tilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$ (see above). If there are some summands of the form $p^k - 1$ in the partition ω' , then $\alpha_{\omega'} \in p\Omega_U(\mathbb{Z}) \otimes \mathbb{Z}_{(p)}$, i.e. $s_\omega(\alpha_{\omega'})$ is divisible by p . Anyway, the whole sum in (12) is divisible by p . The necessity of the theorem is proved.

(b) Sufficiency.

Since all the K -characteristic numbers of σ are in $\mathbb{Z}_{(p)}$, it follows from the Stong-Hattori theorem [CF2] that $\sigma \in \Omega_U \otimes \mathbb{Z}_{(p)}$. Besides, suppose that the characteristic numbers $s_\omega(\sigma)$ are zero modulo p for all divisible by $p-1$ partitions $\omega = \sum_i k_i \cdot (i)$, $\|\omega\| = n$.

Consider again the constructed above generator set $\alpha_1, \alpha_2, \dots$ for $\Omega_U \otimes \mathbb{Z}_{(p)}$. In order to prove that $\sigma \in \tilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$ one needs to show that for every divisible by $p-1$ and non p -adic partition $\omega = \sum_i k_i \cdot (i)$ the coefficient r_ω in decomposition (10) is zero modulo p . Let ω be such a partition. We can rewrite identity (12) as follows:

$$(13) \quad s_\omega(\sigma) = r_\omega s_\omega(\alpha_\omega) + \sum_{\omega' \supset \omega, \omega' \neq \omega} r_{\omega'} s_\omega(\alpha_{\omega'}).$$

One can assume by induction that if a partition ω' such that $\omega' \supset \omega$, $\omega' \neq \omega$, $\|\omega'\| = m(p-1)$, is non p -adic, then the coefficient $r_{\omega'}$ is divisible by p . If the partition $\omega' = \sum_i k'_i \cdot (i)$ is not non p -adic (i.e. there some summands of the form $p^k - 1$), then $s_\omega(\alpha_{\omega'})$ is divisible by p . Anyway, the second summand in the right hand side of (13) is zero modulo p . The left hand side of (13) is zero modulo p by assumption. Since ω is non p -adic, we have $\alpha_\omega = e \cdot \beta_\omega + \dots$ with invertible $e \in \mathbb{Z}_{(p)}$. So, $s_\omega(\alpha_\omega)$ is not divisible by p . Thus, it follows from (13) that r_ω is zero modulo p . \square

Corollary 4.3. *An element $\sigma \in \Omega_U$ is the cobordism class of a manifold with \mathbb{Z}/p -action whose fixed point set has the trivial normal bundle if and only if the cohomological characteristic numbers $s_\omega(\sigma)$, $\|\omega\| = n$, are zero modulo p for all divisible by $p-1$ partitions ω .*

Corollary 4.4. *Each cobordism class of dimension $n \leq 4p-6$ contains a manifold M^n with a simple action of \mathbb{Z}/p .*

This result was firstly proved in [K]. In dimension $n = 4p-4$ there exist a manifold (e.g. $\mathbb{C}P^{2p-2}$) whose cobordism class does not contain a manifold with a simple action of \mathbb{Z}/p .

In Conner and Floyd's book [CF1] it was shown by the methods not involving the formal group theory, that a cobordism class $\sigma \in \Omega_U$ contains a manifold with a *strictly* simple action of \mathbb{Z}/p (see definition 1.1) if and only if *all* the characteristic numbers $s_\omega(\sigma)$ are zero modulo p . More precisely, it was shown there that the set of cobordism classes of manifolds with a strictly simple \mathbb{Z}/p -action coincides with the Ω_U -module spanned by the set $Y^0 = p, Y^1, Y^2, \dots$, where $Y^i \in \Omega_U^{p^i-1}$ are the so-called "Milnor manifolds". These manifolds Y^i are uniquely determined by the following conditions: $s_{(p^i-1)}(Y^i) = p$, and $s_\omega(Y^i)$ is divisible by p for any ω . For our purposes we may consider $\Omega_U \otimes \mathbb{Z}_{(p)}$ -modules instead of Ω_U -modules. Hence, one could take the elements $\alpha_{p^i-1}^{(p)}$ from corollary 3.2 as representatives of the cobordism classes of Y^i . Now, we see that the $\Omega_U \otimes \mathbb{Z}_{(p)}$ -module $\Omega_U[p, Y^1, Y^2, \dots] \otimes \mathbb{Z}_{(p)}$ studied

by Conner and Floyd is included into our $\Omega_U \otimes \mathbb{Z}_{(p)}$ -module $\tilde{\Lambda}_p(1) \otimes \mathbb{Z}_{(p)}$, and the set of generators for the former module is a subset of the generator set for the latter one.

Finally, we note that if a certain cobordism class $\sigma \in \Omega_U$ contains a representative M with a strictly simple action of \mathbb{Z}/p , then it is not necessarily true that *any* simple action of \mathbb{Z}/p on M is strictly simple. Indeed, let us consider two simple actions, first on $M_1 = \mathbb{C}P^{p-1}$ with generator $\rho \in \mathbb{Z}/p$ acting as $\rho(z_1 : \dots : z_p) = (z_1 : \rho z_2 : \dots : \rho^{p-1} z_p)$ (this simple action with p fixed points is strictly simple as well), and second on $M_2 = \mathbb{C}P^1$, $\rho(z_1 : z_2) = (z_1 : \rho z_2)$ (this simple action with 2 fixed points is not strictly simple). Then one has two simple \mathbb{Z}/p -actions on $M = M_1 \times M_2$: $\rho(a, b) = (\rho a, b)$ and $\rho(a, b) = (a, \rho b)$, $a \in \mathbb{C}P^{p-1}$, $b \in \mathbb{C}P^1$. The first one is strictly simple, while the second one is not.

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